**UNIT - II**

**VECTOR DIFFERENTIATION**

**Topic Learning Objectives**:

* Understand the existence of vector functions, derivatives of vector functions and rules of differentiation. Geometrical and physical interpretation of derivative of vector functions.
* The importance of defining vector differential operator and the operations- Gradient of scalar point functions, Divergence and Curl of vector point functions.

**Note:** In all the vectors wherever i, j, k are used they have to be treated as unit vectors along x, y, z directions respectively.

Vector calculus plays an important role in differential geometry and in the study of partial differential equations. Vector calculus originated in the 19th century in connection with the needs of mechanics and physics, when operations on vectors began to be performed directly, without their previous conversion to coordinate form. More advanced studies of the properties of mathematical and physical objects which are invariant with respect to the choice of coordinate systems led to a generalization of vector calculus. It is used extensively in physics and engineering, especially in the description of electromagnetic fields, gravitational fields and fluid flow.

**Vector Fields:**

If at each point (*x*, *y*, *z*) there is an associated vector

, then  is a vector function and the field processing such a vector function is called a vector field.

# Examples:

(i) A magnetic field *B* in a region of space, 



(ii) The velocity field of water flowing in a pipe,.



Vector function is a [function](https://en.wikipedia.org/wiki/Function_(mathematics)) whose domain is set of real numbers and whose range is a set of vectors.

**Differentiation of a Vector Function:**

Let the position vector of a point P (x, y, z) in space be   .

If *x, y, z* are all functions of *t*, then ***r*** is said to be a vector function of *t*. As the parameter *t* varies the point P traces a curve in space. Therefore (t) = x(t)+ y(t)j + z(t) k is the vector equation of the curve, where *x(t), y(t)*and *z(t)* are real functions of the real variable *t*.

This function can be viewed as describing a space curve. Intuitively it can be regarded as a position vector, expressed as a function of ‘*t*’ that traces out a space curve with increasing values of *t*.

If  is a vector function of a scalar variable *t* then the derivative of with respect to *t* is



• For example, suppose you were driving along a wiggly



road with position *r (t)* at time *t*.

• Differentiating *r (t)* should give velocity *v (t).*

• Differentiating *v (t)* should yield acceleration *a (t)*.

• Differentiating *a (t)* should yield the jerk *j (t).*

# Velocity and Acceleration:

If *(t) = x(t)+ y(t) j + z(t) k* is the position vector of a particle moving along a smooth curve in space, then is the particle’s **velocity vector**, tangent to the curve. At any time *t*, the direction of ***v(t)*** is the **direction of motion**, the magnitude of ***v(t)*** is the particle’s **speed**, and the derivative , when it exists, is the particle’s **acceleration vector**.

In summary,

* Velocity is the derivative of position vector:
* Speed is the magnitude of velocity:
* Acceleration is the derivative of velocity:
* Unit Tangent vector is the direction of motion at time *t*.
* Component of velocity along a given vector is
* Component of acceleration along a given vector is

**Differentiation rules for vector functions:**



**Examples:**

# 1. A particle moves such that its position vector at time *t* is .

Determine its velocity, acceleration and their magnitude, direction at time *t* = 0.

**Solution:** velocity : 

, magnitude = , direction is

acceleration: , , magnitude =, direction is .

**2.** For the curves whose equations are given below, find the unit tangent vectors:

(i)  at .

(ii)  at 

**Solution:** (i) In the vector form equation of the given curve is







 Unit tangent vector to the given curve at a point ‘’ is given by



At t = 0, 

(ii) 







At t=, =

**3.** Find the angle between the tangents to the curve **** at the points .

**Solution:** 





At 

At 

Angle between unit tangent vectors at the points  is given by



.

**4.** A particle moves along the curve ,  where  is a constant. Find  so that acceleration is perpendicular to position vectors at .

**Solution:** At time t position vector of particle is







Acceleration =

, 

Given acceleration is perpendicular to position vector,



**5.** A particle moves along the curve . Find the component of velocity and acceleration in the direction of vector at .

**Solution:** Given 

Velocity 

Acceleration=

At , 



Also 





 Component of velocity at along the given vector is,



Component of acceleration atalong the given vector is,

.

**Exercise:**

1. A person on a hang glider is spiralling upward due to rapidly rising air on a path having position vector Find

(a) The velocity and acceleration vectors

(b) The glider’s speed at any time *t.*

2. Given the curve find the unit tangent vector at the point

**Answers**

1. ;

2.

**Scalar and Vector Point Functions:**

A physical quantity that can be expressed as a continuous function and which can assume definite values at each point of a region of space is called a point function in that region, and the region containing the point function is called a field.

There are two types of point functions namely **scalar point function** and **vector point function**.

**Scalar point function**:

At each point  of a region R in space if there corresponds a definite scalar , then such a function  is called a scalar point function and the region is called a scalar field.

**Examples:** Functions representing the temperature, density of a body, gravitational potential etc. are scalar point functions.

**Vector point function**: At each point  of a region R in space if there corresponds a definite vector, then such a function is called a vector point function, the region is called a vector field.

**Examples:** Functions representing the velocity of moving fluid particle, gravitational force, etc. are vector point functions.

**Level surface**: The scalar point function  is usually called the potential function and  represents the family of surfaces in the scalar field. If at each point on the surface,= c has the same value then the surface is called the level surface.

**Definition**: The vector differential operator denoted by read as **del** or **nabla** is defined by  is called **vector differential operator**. This operator has no meaning on itself but assumes specific meaning depending on how it operates on a scalar or vector point function.

**Gradient of a scalar point function**: Let be any scalar point function defined at some point  of a scalar field so that the function is continuously differentiable. Then the vector function  is called a gradient of scalar function  and it is denoted by  or grad. Thus .

**Note**:

1. If  is a scalar point function, then  are called components of grad

2. is called the magnitude of grad.

**Geometrical interpretation of gradient:** is a vector normal to the surface and has a magnitude equal to the rate of change of *f* along this normal.

**Properties of Gradient**

1. The differential  of is given by  where 

2. For any scalar function  and and any scalar and  

3. For any scalar function  and 

i)  ii)  if 

**Unit normal vector:**

Since  is normal vector to surface  then unit vector is denoted by  and is defined as,  where = normal vector.

**Note:** The angle between the normal’s to the surfaces is given by .

**Directional derivative:**

If is any vector incline at an angle  to the direction of  where is scalar point function then

.

It represents component of  in the direction of which is known as directional derivative of  in the direction of .

**Maximum Directional Derivative:**

The direction derivative will be maximum in the direction of   and maximum value of the directional derivative 

Maximum directional derivative is also called normal derivative.

 normal derivative 

**Problems:**

**1.** If  then find  and  at (1, -1, 1).

**Solution**: Given 





At (1, -1, 1), 

.

**2.** Find the directional derivative of  at the point

(1, -2, 1) in the direction of .

**Solution:** 



At (1, -2, 1), 

Given

The directional derivative of  is

The directional derivative.

**3.** Find the directional derivative of  at ( 2, -1, 1) in the direction .

**Solution**: 



At (2, -1, 1) 

Given

The directional derivative of  is

The directional derivative.

**4.** Find the directional derivative of  at the point

(-1, 2, 3) in the direction towards the point (2, -1, -1).

**Solution**: Given 



At (-1, 2, 3) 

Let  and 

The directional derivative is

The directional derivative.

**5.** Find the directional derivative of  at the point  in the direction of .

**Solution**: Let 



At the point 

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The directional derivative

The directional derivative.

**6.** Find the maximum directional derivative of at .

**Solution**: Given 

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The maximum directional derivative =

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**7.** Find the maximum directional derivative of at **.**

**Solution**: Given 



At 

The maximum directional derivative =  = 

**8.** Find the unit normal vector to the surface at (1, 0, 2).

**Solution**: Let 



At (1, 0, 2) 



The unit normal vector 

**9.** Find the unit normal vector to the surface  at the

point (1, -1, 2).

**Solution**: 

At (1, -1, 2)

 and 

The unit normal vector 

**10.** Find the angle between the normals to the surface  at the points (2, -2, 4) and (-1, -1, 1).

**Solution**: Let 





Now at (2, -2, 4), 

At (-1, -1, 1), 

Unit normal vector to the surface at (2, -2, 4) is 

Unit normal vector to the surface at (-1, -1, 1) is 

Angle between the normals is given by 



 is the angle between the normals.

**11.** Find the angle between the normals to the surface  at the points

(1, 9, -3) and (-2, -2, 2).

**Solution**: Let 





Now at (1, 9,-3) 



At (-2, -2, 2), 



Angle between the normal is 



Hence the acute angle .

**12.** Find the angle between the surfaces  and  at the point (2, -1,2) common to them.

**Solution**: The angle between the two surfaces at common point is angle between the normals drawn to the surfaces at that point.

Let , 

At (2, -1, 2) 

Now 

Let , 

At (2, -1, 2) 

Now 

Angle between the normals is 





**13.** Find whether the surfaces  and  intersect orthogonally at the point.

**Solution**: Let k

At 



At (1, -1, -2), 



Angle between two normals is 





Therefore the surfaces intersect orthogonally.

**14.** Find the constants and so that the surface  is orthogonal to the surface  at the point (-1, 2, 1).

**Solution**: Let 



At (-1, 2, 1) 

Now 

, 

At (-1, 2, 1) 



Since the surfaces intersect orthogonally





i.e............... (1)

Also the point (-1, 2, 1) lies on the surface 

i.e. .............. (2)

Solving the equation (1) and (2)  and 

**Exercise:**

1. If  then find  at .

2. Find the directional derivative of  at (1, 2,-1) in the direction of

.

3. Find the maximum directional derivative of at the point

(1, -2, 3).

4. Find the unit normal vector to the surface  at (1, -1, 1).

5. Find the unit normal vector to the surface at (1, -1, 2).

6. Find the angle between the normals to the surface  at the points

(4, 1, 2) and (3, 3, -3).

7. Find the angle between the normals to the surface at the points

(2, -2, 4) and (-1, -1, 1).

8. Find the angle between the normals to the surface  at the points

(1, 1, 1) and (2, 1, 1).

9. Find the angle between the normals to the surface  and

 at the point (1, 1, 1) common to them.

10. Find the constants and so that the surfaces  and

intersect orthogonally at the point (1, 1,-2).

**Answers:**

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**Divergence of a vector function:**

Let be a continuously differentiable vector function, then divergence of a vector point function is denoted by or and defined as

or

Clearly divergence of a vector point function is a scalar point function.

**Physical interpretation**: If represents a velocity field of a gas or fluid then represents the **rate of expansion per unit volume under the flow of gas or fluid**.

**Definition:** A vector function  is said to be a **Solenoidal** if .

Clearly constant vector function is a solenoidal vector function.

**Curl of vector function**: Let be a continuously differentiable vector function, then operating vectorially on is denoted by or is given by

Clearly curl of a vector function is a vector function.

**Physical interpretation**: The curl of a vector function represents **rotational motion.**

**Definition**: A vector function  is said to be irrotational vector function if .

**Laplacian of a scalar field**

Let  be a given scalar field. Then is a vector field given by,



 divergence of  is given by





The RHS is call Laplacian of and denoted by .

 By definition  …. (1)



Equation (1) can be rewritten as,



 is the differential operator given by,  and is called Laplacian operator.

**Problems:**

**1.** If  then find .

**Solution**: 

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**2.** If  then find at (1, 2, 0).

**Solution:** 

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At (1, 2, 0).

**3.** If  then find .

**Solution**: 



**4.** If  then find .

**Solution**: 





**5.** If  and  then find .

**Solution**:

**6.** Show that the vector function  is solenoidal.

**Solution**: Consider 



is solenoidal vector function.

**7.** If is solenoidal vector field, then find the value of

**Solution**: If  is solenoidal then

Hence 



**8.** If  and  then show that

**Solution:** Given , =

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**9.** If  and  then show that  is solenoidal.

**Solution:** Given , =

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Consider 





is solenoidal vector field.

**10.** If  and  then prove that .

**Solution**: Given , =

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**11.** If  and  then prove that .

**Solution**: Given , =



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**12.** If  and  then prove that and

hence show that .

**Solution**: From the problem number 12 we already proved that

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**13.** If  then find .

**Solution**: 

**14.** Show that  is irrotational.

**Solution**: Consider 

is irrotational.

**15.** If  then find .

**Solution:** 



**16.** Prove that 

Let 



is irrotational.

**17.** For any differentiable vector function  prove that .

**Solution**: Let 







But mixed partial derivatives are equal.

 is solenoidal.

**18.** For what value of the vector field  is irrotational.

**Solution:** If is irrotational, then 

Hence 





**19.** Find the constants so that

 is irrotational.

**Solution**: If is irrotational, then 

Hence 





**20.** Show that  is irrotational and find the function such

that .

.**Solution:**  Given that 

= 



is irrotational vector field.

We have to find the function  such that

i.e.



We have 



Regrouping the terms we get

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**21.** Show that  is irrotational. Find the function  such that .

**Solution**: Given that 

Consider 



Therefore  is irrotational.

Find the function  such that 









Regrouping the terms

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**22.** Show that  is irrotational. Find the function  such that .

**Solution**: Given 

Consider 



is irrotational.

Find the function  such that

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Regrouping the terms







**23.** If  and  then show that  is irrotational for all values of and solenoidal for

**Solution**: Given , =

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is solenoidal implies , .

**Exercise**:

1. If  then find  at (1, 2, 3)
2. If  then find 
3.  then find and 
4. Show that the vector field  is solenoidal.
5. Show that  is solenoidal.
6. Determine so that the vector field  is solenoidal.
7. Determine the constant such that the vector field

is solenoidal.

1. If  then find 
2. Show that  is irrotational.

10. Show that  is irrotational.

11. Show that the vector field  is irrotational.

12. If  and  then show that  is solenoidal.

13. If is irrotational vector

Field, then find the constants.

14. If  then show that  is irrotational.

15. If  then show that  satisfies the Laplacian equation.

16. If  then find  at (1, 1, 1).

17. If  then find .

18. If  and  then find 

19. If  and  then prove that  is solenoid.

20. If  and  then show that 

21. If  and  then find  and hence show that



22. If  where  is a constant vector. Then prove that 

23. Prove that  is irrotational and  is solenoidal.

24. Show that  is irrotational and find the function 

such that .

25. Show that  is

irrotational and find the function  such that 

26. Show that  is irrotational and find the function  such that 

27. Show that where 

28. Find the scalar function  such that  given

that at the origin.

29. Show that 

30. Show that  is both solenoidal and irrotational.

**Answers:**

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**Video Links:**

**1.** Vector differentiation

<https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives/position-vector-functions/v/differential-of-a-vector-valued-function>

**2.** Gradient

<https://www.youtube.com/watch?v=fZ231k3zsAA>

<https://www.youtube.com/watch?v=GkB4vW16QHI>

**3.** Directional derivative

<https://www.youtube.com/watch?v=Dcnj1bYEZlY>

**4.** Applications of Gradient, Divergence and curl

<https://www.youtube.com/watch?v=qOcFJKQPZfo> <https://www.youtube.com/watch?v=vvzTEbp9lrc>

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